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PHYSICAL CONTENT OF A GAUGE MODEL DESCRIBING MEDIA WITH STRUCTURE AND DEFECTS

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The possibility of a unified description of media with structure and media with defects is established on the basis of the relation between the main geometrical concepts and the dynamic equations of the conventional theories of elastic media with microstructure and the continual theory of defects. An idealized scheme of representing the general deformation in a material with defects is proposed.

Introduction. The development of adequate models for describing the inelastic behavior of materials is topical for theoretical predictions of the behavior of materials under various actions. According to modern concepts, the inelastic deformation of solids is substantially nonhomogeneous, i.e., a medium being deformed is a set of regions with different nature and degree of deformation. A region of homogeneous deformation is an individual element of the structure. In this sense, all real solids being deformed are media with structure whose scale is determined by various structural peculiarities of the medium, for example, by the distribution of stress concentrators. The presence of structure implies the existence of defects in the form of interfaces between structural elements. Traditionally, some aspects of inelastic behavior due to the structure and defects of materials have been considered separately in the continual theory of defects [1, 2] and the theories of elastic media with structure [3, 4]. Until recently, the continual theory of defects did not contain dynamics. This approach allowed one to calculate strain and stress fields at a specified density of defects. The gauge field theories [5, 6] has made it possible to derive a closed system of dynamic equations for an elastic body with defects. Within the framework of the gauge approach, media with structure have not been described in the literature, and this makes the present paper topical.

1. Mathematical Formalism in Constructing the Gauge Theory. Gauge theories were first used to describe the deformation of solids with defects in [5, 6], where the mathematical apparatus of gauge description is discussed in detail and it is shown that dynamic models of an elastic body with dislocations, disclinations, and defects of both types can be developed on the basis of a Lagrangian nonlinear elastic body. Below, as a first approximation we consider a linear model of an elastic body with dislocations. The procedure of constructing the gauge model is as follows: we write the Lagrangian of the linear theory of elasticity for a homogeneous isotropic body in the form

$$L = \int dV \Big[\frac{\rho}{2}\frac{\partial u_i}{\partial t}\frac{\partial u_i}{\partial t} - \frac{\mu}{2}\Big(\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\Big) - \frac{\lambda}{2}\frac{\partial u_i}{\partial x_i}\frac{\partial u_j}{\partial x_j}\Big] \tag{1.1}$$

(u_i are components of the elastic displacement vector, λ and μ are Lamé coefficients, and ρ is the density of the medium) and determine its gauge group. The initial Lagrangian (1.1) is invariant with respect to homogeneous translations

$$\tilde{u}_i(x,t) = u_i(x,t) + a_i, \tag{1.2}$$

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and this corresponds to the displacement of an elastic body as whole. The localization of the translation group

$$\tilde{u}_i(x,t) = u_i(x,t) + a_i(x,t) \tag{1.3}$$

violates the invariance of (1.1) since additional terms appear due to differentiation of the parameters of the group $a_i(x,t)$. The procedure of minimum replacement, in which ordinary derivatives are replaced by elongated derivatives,

$$\frac{\partial \tilde{u}_i}{\partial x_j} \to D_j u_i = \frac{\partial u_i}{\partial x_j} + \beta_{ji}, \qquad \frac{\partial \tilde{u}_i}{\partial t} \to D_0 u_i = \frac{\partial u_i}{\partial t} + V_i, \tag{1.4}$$

restores the invariance of (1.1) for the inhomogeneous transformations (1.3):

$$L_1 = \int dV \Big[\frac{\rho}{2}\, D_0 u_i D_0 u_i - \frac{\mu}{2}\,(D_j u_i D_j u_i + D_j u_i D_i u_j) - \frac{\lambda}{2}\, D_i u_i D_j u_j\Big]. \tag{1.5}$$

The replacement (1.4) gives rise to new gauge or compensating fields β_{ji} and V_i, with additional, according to [5], kinetic and potential energy, which determines the Lagrangian of the gauge fields

$$L_2 = \int dV \Big[\frac{B}{2}\, I_{ij} I_{ij} - \frac{S}{2}\, \alpha_{ij}\alpha_{ij}\Big] \tag{1.6}$$

as a function of the quantities

$$I_{ij} = \frac{\partial V_j}{\partial x_i} - \frac{\partial \beta_{ij}}{\partial t}, \qquad \alpha_{ij} = e_{ikl}\,\frac{\partial \beta_{lj}}{\partial x_k}, \tag{1.7}$$

where B and S are new constants of the theory. The procedure of constructing the gauge theory does not clarify the physical content of the model. To solve this problem, we analyze the general deformation within the framework of the continual theory of defects, since the localization of the translation group (1.3) at the point (x,t) is equivalent to introducing a Volterra single dislocation, and the functional dependence on the coordinates gives the averaged distribution of defects [2, 7].

2. Scheme of Representation of General Deformation in the Continual Theory of Defects. The general deformation in a material with defects can be written as the sum of the following three terms known in the continual theory of defects:

$$u^{\text{tot}}_{(i,j)} = u^{\text{el}}_{(i,j)} + u^{\text{el-pl.D}}_{(i,j)} + u^{\text{c.pl}}_{(i,j)}, \tag{2.1}$$

each of which represents the symmetric part of the gradient of the continuous displacement vector. The subscripts in parentheses denote symmetrization, and the comma denotes differentiation with respect to the coordinate. The first term on the right side of (2.1) corresponds to the elastic deformation produced by external loads and it disappears with acoustic wave velocity when the loads are removed. The second term describes the compatible elastoplastic deformation due to defects of the material. As adopted in the continual dislocation theory [1, 2], the gradient $u^{\text{el-pl.D}}$ is the sum of the elastic and plastic distortions

$$u^{\text{el-pl.D}}_{i,j} = \beta^{\text{el.D}}_{ji} + \beta^{\text{pl.D}}_{ji}, \tag{2.2}$$

each of which is not a gradient of the continuous displacement vector. By definition, arbitrarily specified plastic distortion $\beta^{\text{pl.D}}_{ji}$ corresponds to dislocation density $\alpha_{ij} = -e_{ikl}\,\partial\beta^{\text{pl.D}}_{lj}/\partial x_k$, and elastic distortion $\beta^{\text{el.D}}_{ji}$ describes distortions of the body that ensure its continuity at the specified dislocation density:

$$e_{ikl}\,\frac{\partial}{\partial x_k}\, u^{\text{el-pl.D}}_{l,j} = e_{ikl}\,\frac{\partial}{\partial x_k}\,(\beta^{\text{el.D}}_{lj} + \beta^{\text{pl.D}}_{lj}).$$

Hence, the dislocation density can be written as

$$\alpha_{ij} = e_{ikl}\,\frac{\partial \beta^{\text{el.D}}_{lj}}{\partial x_k}. \tag{2.3}$$

Since, separately, $\beta^{\text{el.D}}_{ji}$ and $\beta^{\text{pl.D}}_{ji}$ do not satisfy compatibility conditions, they are called the incompatible elastic distortion and the incompatible plastic distortion, respectively. The last term in (2.1) corresponds to the compatible plastic deformation that is not related to stresses and it describes irreversible changes in the shape of the body, for example, due to annihilation of defects or their emergence on the surface.

3. Meaning of the Variables of the Gauge Model. The proposed idealized representation of the general deformation (2.1), which is unquestionably expedient for an analysis of the main mechanical characteristics of solids, shows that the elastic distortions in a material with defects (1.4) are determined by the reversible elastic distortion due to external loads and the incompatible elastic distortion due to defects of the material: $D_j u_i = \partial u_i^{\rm el}/\partial x_j + \beta_{ji}^{\rm el.D}$.

The rate of displacements, which determines the kinetic energy L_1, can be represented as the sum of the rates of elastic displacements and the displacements due to motion of defects: $D_0 u_i = \partial u_i^{\rm el}/\partial t + V_i$, where $V_i(x,t) = (\partial/\partial t) u_i^{\rm el\text{-}pl.D}(x,t) + f_i(t)$ [$f_i(t)$ is an unknown function of time]. The last expression, illustrating the physical content of the potential V_i, should not be substituted into formula (1.4) since it increases the number of variables of the model and the order of equations of motion. In addition, the processes due to motion of defects are substantially dissipative and determine the viscous properties of materials, which are described using the rate as an independent variable.

For known values of the potentials $u_i^{\rm el}$, $\beta_{ji}^{\rm el.D}$, and V_i, the Lagrangian of the gauge fields L_2 is determined by the dislocation density tensor (2.3) and the dislocation flux density tensor I_{ij}. This quantity is found as the time derivative of the plastic distortion due to defects of the material [7]. By simple transformations with allowance for Eq. (2.2), it can be expressed in terms of the incompatible elastic distortion and the rate V_i due to motion of defects:

$$I_{ij} = -\frac{\partial \beta_{ij}^{\rm pl.D}}{\partial t} = \frac{\partial V_j}{\partial x_i} - \frac{\partial \beta_{ij}^{\rm el.D}}{\partial t}.$$

The condition of stationarity of the action integral

$$\delta_\theta A = \int dt \int\limits_V \Big[\Big(\frac{\partial L}{\partial \theta} - \frac{\partial}{\partial x_i}\Big(\frac{\partial L}{\partial \theta_{,i}}\Big)\Big)\delta\theta\Big] dV + \int dt \int\limits_S \Big[\Big(\frac{\partial L}{\partial \theta_{,i}}\Big)\delta\theta\Big] dS = 0,$$

where θ takes the values $u_i^{\rm el}$, $\beta_{ji}^{\rm el.D}$, and V_i, leads to the dynamic equations of the model:

$$\frac{\partial}{\partial t}\rho\Big(\frac{\partial u_i}{\partial t} + V_i\Big) = \frac{\partial \sigma_{ji}}{\partial x_j}, \qquad \frac{\partial}{\partial t}B\Big(\frac{\partial \beta_{ij}}{\partial t} - \frac{\partial V_j}{\partial x_i}\Big) - S\frac{\partial}{\partial x_p}\Big(\frac{\partial \beta_{ij}}{\partial x_p} - \frac{\partial \beta_{pj}}{\partial x_i}\Big) = \sigma_{ij},$$
$$\frac{\partial}{\partial x_i}B\Big(\frac{\partial \beta_{ij}}{\partial t} - \frac{\partial V_j}{\partial x_i}\Big) = \rho\Big(\frac{\partial u_j}{\partial t} + V_j\Big) \tag{3.1}$$

and the conditions on the surface of the body, which are satisfied for

$$\delta u_i\Big|_S = 0, \qquad \delta\beta_{ij}\Big|_S = 0, \qquad \delta V_i\Big|_S = 0 \tag{3.2}$$

or for

$$\sigma_{ij}\Big|_S = 0, \qquad \alpha_{ij}\Big|_S = 0, \qquad I_{ij}\Big|_S = 0 \tag{3.3}$$

and are a consequence of the equations

$$\frac{\partial L}{\partial \theta} - \frac{\partial}{\partial x_i}\frac{\partial L}{\partial \theta_{,i}} = 0, \qquad \frac{\partial L}{\partial \theta_{,i}}\delta\theta\Big|_S = 0.$$

The superscripts of the elastic quantities $u_i^{\rm el}$ and $\beta_{ji}^{\rm el.D}$ in expressions (3.1)–(3.3) and in the further text are omitted. Zero variations on the boundary (3.2) correspond to specified values of the quantities and determine first-order boundary conditions. Equalities (3.3) represent the boundary conditions on the free surface, where α_{ij} is the dislocation density tensor, I_{ij} is the dislocation flux tensor, and σ_{ij} are effective stresses defined by $\sigma_{ij} = (\mu/2)(u_{i,j} + \beta_{ji} + u_{j,i} + \beta_{ij}) + (\lambda/2)(u_{i,i} + \beta_{ii})\delta_{ij}$. In solving problem with specified actions on the boundary, one should add corresponding surface terms in the expression of the Lagrangian density (1.1).

Since the Lagrangian of the model is invariant with respect to covariant derivatives, the first of the three groups of equations of motions (3.1)–(3.3) is a consequence of the two other. Selecting the relation that permits one to eliminate one variable from the two other equations (gauge condition), it is possible to find a solution of the closed system of equations for two variables subject to specified initial and boundary conditions that distinguish unambiguous solutions. Under the condition $V_i = 0$, which implies that the flow of defects does not cause motion of the material continuum, the dynamic equations

$$\rho \frac{\partial^2 u_i}{\partial t^2} - \frac{\partial \sigma_{ji}}{\partial x_j} = 0, \qquad B \frac{\partial^2 \beta_{ij}}{\partial t^2} - S \frac{\partial}{\partial x_p}\Big(\frac{\partial \beta_{ij}}{\partial x_p} - \frac{\partial \beta_{pj}}{\partial x_i}\Big) - \sigma_{ij} = 0 \tag{3.4}$$

with the corresponding initial and boundary conditions represent a dynamic model of an elastic body with internal stresses. The system obtained can be used to analyze the elastoplastic material behavior, which generally implies the existence of three deformation components (2.1) provided that the compatible plastic deformation is insignificant. These conditions are realized in shock-wave loading processes, in which most defects do not emerge on the surface and give rise to compatible plastic deformation.

4. Relationship of the Gauge Theory with the Theories of Micromechanics. To justify the applicability of the rigorous mathematical formalism of the gauge theories to the deformation of media with structure, we consider the relationship of the gauge theories with the theories of micromechanics [3, 4]. In the theories of micromechanics, each point represents a volume that is not absolutely rigid and undergoes some deformation. The radius vector of an arbitrary point in a medium with structure is the sum of the two vectors $\boldsymbol{R}(r, r', t) = \boldsymbol{R}(r, t) + \boldsymbol{R}'(r, r', t)$, where $\boldsymbol{R}(r, t)$ determines the location of the center of mass of a structural element and $\boldsymbol{R}'(r, r', t)$ is the position of the selecting point relative to a coordinate system attached to the center of mass of the structural element. Using macrolevel coordinates, the vector $\boldsymbol{R}(r, r', t)$ can be averaged over the volume of the structural element, which gives $\boldsymbol{R}(r, l, t) = \boldsymbol{R}(r, t) + \boldsymbol{R}'(r, l, t)$ or, going over to the displacement vector, $\boldsymbol{u}(r, l, t) = \boldsymbol{u}(r, t) + \boldsymbol{u}'(r, l, t)$, where l is the linear dimension of the structural element. This expression implies that the displacement of a macropoint in a medium with structure is the average displacement of points of the structural element, determined by the displacement of the center of mass of the structural element and the mean projection of the relative displacements of points of the structural element onto the macrocoordinates. The vector $\boldsymbol{u}'(r, l, t)$ is a peculiar response of the mesolevel to the macrolevel that shows how the displacement of the macropoint changes upon deformation of the corresponding volume. In the gauge theories, this expression corresponds to localization of the translation group (1.3) and determines a defect of the translation type. From the previous reasoning it follows that the localization of the translation group is due to manifestation of the internal structure of the macropoint and is a point that unifies the continual theory of defects and the theory of elastic media with structure.

In the theories of micromechanics, the deformation geometry is developed on the basis of the assumption of the affine deformation of the structural $u'_i(r, r', t) = r'_k \varphi_{ki}(r, t)$, which implies that the gradient of relative displacements (microdistortion) is homogeneous in a structural element and is nonhomogeneous in the macrovolume. An integral of $\varphi(r, t)$ over the macrocontour, whose points are structural elements, is not equal to zero point and determines the jump of displacements, which is associated with the Burgers vector in the theory of defects. This also indicates the common nature of the deformation of media with structure and media with defects.

The dynamic equations in the theories of micromechanics are determined on the basis of two approaches: by postulating the Lagrangian of a medium with structure [3] or local conservation laws [4]. These approaches use appropriate boundary conditions. The dynamic equations of Mindlin's theory, developed on the basis of the Lagrangian of a medium with structure [3], coincide with accuracy up to coefficients with the equations of motion in the gauge model (3.4). The coincidence of the dynamic equations again demonstrates the possibility of describing media with structure within the framework of the gauge theory and allows one to relate the unknown constants of the gauge model B and S to the inertial properties of structural elements and the double stress moduli of Mindlin's theory [8]. Thus, the relationship between the quantities included in the dynamic equation of the theories and boundary conditions is established.

Conclusion. By rigorous generalization of the classical theory of elasticity within the framework of the gauge formalism (1.1)–(1.7), it is possible to develop a dynamic theory of deformation for solids that is a consistent extension of the continual theory of defects and a version of the theory of elastic media with structure. From the viewpoint of thermodynamics, the applicability of the gauge approach to the deformation of media with structure can be substantiated as follows. The initial Lagrangian of the classical theory of elasticity (1.1) describes irreversible equilibrium deformation processes in the elastic region. Deformation beyond the elastic limit is an irreversible nonequilibrium process, and, hence, the variational formulation of the problem on the basis of the Lagrangian (1.1) becomes invalid. Deformation beyond the elastic limit can be described on the basis of the local equilibrium principle, whose essence is that any nonequilibrium state of a body can be regarded as a set of equilibrium states of small volumes making up the body. In other words, the material acquires structure whose elements are small volumes in the state of internal mechanical equilibrium. Since the only equilibrium process in the mechanics of a deformable solid is elastic deformation, the small volumes, undergoing elastic deformation, can be displaced, without a change in the internal equilibrium state, as a unit by a certain vector $\boldsymbol{a}(x,t)$, whose coordinate dependence distinguishes the volume element considered. Thus, from the viewpoint of thermodynamics, transition from the global gauge transformations (1.2) to the local transformation (1.3) implies the existence of regions of local mechanical equilibrium, which can be considered as separate elements of the structure.

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